Reduced Magnetohydrodynamic Theory of Oblique Plasmoid Instabilities

Scott Baalrud, Amitava Bhattacharjee, and Yi-Min Huang

Center for Integrated Computation and Analysis of Reconnection and Turbulence (CICART)

University of New Hampshire

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Outline

• Use boundary layer theory to derive the plasmoid dispersion relation including oblique angles from linear RMHD for Harris equilibrium

• Compare results of BL theory with direct numerical solutions

• Angle of obliquity: $\theta = \arctan(k_z/k_y)$, $\hat{z}$ guide field direction, $\hat{y}$ sheared field direction

• Main results:
  -> Resonant point shifts: $x_s = -\lambda\arctanh[k_z B_{zo}/(k_y B_{yo})]$
  -> Instability requires $|\theta| < \arctan(\bar{B}_{yo}/B_{zo})$
  -> Most unstable angle is oblique ($\theta \neq 0$) in the constant-$\psi$ branch
  -> Most unstable angle is parallel ($\theta = 0$) in the nonconstant-$\psi$ branch
  -> Most unstable wavenumber lies at intersection of two branches, and is a parallel mode
  -> Magnetic island topology changes for oblique modes
Motivated by Collisionless Kinetic Theory

- Daughton et al. Nature Phys. 7, 539 (2011), show discrepancy between collisionless kinetic theory and linear Vlasov simulations for oblique modes
- Suggested that this is a breakdown of asymptotic boundary layer analysis

**Figure 2 | Theoretical predictions for oblique tearing instability.** The asymptotic theory (solid) from equation (1) is compared with the exact Vlasov results (dashed) as a function of oblique angle $\theta = \tan^{-1}(k_y/k_x)$ for the mass ratio $m_i/m_e$ and sheet thickness $\lambda$ indicated. **a-c**, The growth rate; **d**, the real frequency corresponding to **c**. Other parameters are held fixed, $k\lambda = 0.4, B_{y0} = B_{x0}, T_i = T_e$ and $n_b = 0.3n_0$. 

**CMSO meeting, October 19, 2011, p 3**
We First Consider the MHD Problem

- Heuristic sketch of constant flux surfaces: (a) $\theta = 0$, (b) $\theta \neq 0$

- Oblique plasmoids are similar to $n > 0$ tearing modes in a tokamak

- Two main differences:
  1. Current distributions: smooth (tokamak), current sheet (plasmoid)
     - Consequences for scaling with resistivity
  2. Boundary conditions: periodic (tokamak), open (plasmoid)
     - Consequences for the possible unstable modes ($n$ and $m$ number for tokamak, $k$ and $\theta$ for plasmoid)
Use the Linearized Reduced MHD Equations

- Reduced MHD equations [Strauss, Phys. Fluids 19, 134 (1976)]:

\[
\begin{align*}
\partial_t \Omega + [\Omega, \phi] &= [J_z, \psi] + B_z \partial_z J_z \\
\partial_t \psi &= B_z \partial_z \phi + [\phi, \psi] + S^{-1} \nabla^2 \psi + E_o
\end{align*}
\]

Vorticity: \( \Omega \equiv -\nabla^2 \psi \)

Current along \( \hat{z} \): \( J_z = -\nabla^2 \psi \)

Perpendicular gradient: \( \vec{\nabla}_\perp = \partial_x \hat{x} + \partial_y \hat{y} \)

Poisson bracket: \( [f, g] = (\nabla f \times \nabla g) \cdot \hat{z} \)

- Linearize equations: \( f = f_o + f_1(x) \exp[i(k_y y + k_z z) + \gamma t] \)

\[
\gamma(\phi_1'' - k_y^2 \phi_1) = iF(\psi_1'' - k_y^2 \psi_1) - iF'' \psi_1
\]

and

\[
\gamma \psi_1 = iF \phi_1 + S^{-1}(\psi_1'' - k_y^2 \psi_1)
\]

- Have assumed \( \partial_x \phi_o, \partial_y \phi_o \ll \gamma \), and

\[
F \equiv \vec{k} \cdot \vec{B}_o
\]

- Will use \( k \) and \( \theta \) instead of \( k_y \) and \( k_z \):

\[
k = |k_y|[1 + \mathcal{O}(\epsilon)] \quad \text{and} \quad k_z/k_y = \arctan(\theta) \simeq \theta + \mathcal{O}(\epsilon^3)
\]
Boundary layer analysis: Outer region

- In the outer region, we assume $S^{-1} \ll \gamma \ll 1$
- RMHD equations reduce to the ideal MHD force balance

$$\psi_1'' - (k^2 + F''/F)\psi_1 = 0$$

- Match inner and outer regions via the tearing stability index\(^1\)

$$\Delta' \equiv \lim_{\varepsilon \to 0} \frac{1}{\psi_1(x_s)} \left( \frac{d\psi_1}{dx} \bigg|_{x_s+\varepsilon} - \frac{d\psi_1}{dx} \bigg|_{x_s-\varepsilon} \right)$$

$x_s$ is the resonance point: $F(x = x_s) = 0$

- (1) In the large $k$ limit: $\Delta' \to -2k$
- (2) In the small $k$ limit:

$$\Delta' \to \frac{[F'(x_s)]^2}{k} \left( \frac{1}{F_{-\infty}^2} + \frac{1}{F_{\infty}^2} \right)$$

- Get a good approximation by adding the asymptotic solutions

$$\Delta' \sim \frac{[F'(x = x_s)]^2}{k} \left( \frac{1}{F_{-\infty}^2} + \frac{1}{F_{\infty}^2} \right) - 2k$$

\( \Delta' \) for Harris Equilibrium

- We concentrate on the Harris current sheet with guide field
  \[ \vec{B}_o = \vec{B}_{oy} \tanh(x/\lambda) + B_{oz} \]

- For this, the resonant point is
  \[ x_s = -\lambda \arctanh(\mu) \quad \text{where} \quad \mu \equiv \frac{k_z B_{zo}}{k_y \vec{B}_{yo}} \simeq \theta \frac{B_{zo}}{\vec{B}_{yo}} \]

- Existence of a resonant point requires: \( |\theta| \lesssim \vec{B}_{yo}/B_{zo} \)

- At the resonant point: \( F'(x = x_s) = k \vec{B}_{oy}(1 - \mu^2) \)

- Our \( \Delta' \) expression then gives
  \[
  \Delta'_H \simeq 2 \left( \frac{1 + \mu^2}{k} - k \right)
  
  \]

- Daughton et al. have proposed a different solution\(^2\)
  \[
  \Delta'_D \simeq 2 \left( \frac{1}{k} - k \right) \left[ 1 + \mu^2 \frac{(1 - k/2)}{1 - k} \right]
  
  \]

- Both solutions are exact for normal modes \( \theta = \mu = 0 \)

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Approximate Solution is Accurate for all Angles

- Numerical solution from ideal MHD force balance with Harris CS
- Both theories asymptote to $\Delta' \rightarrow 2(1 + \mu^2)/k$ for small $k$
- Possible to choose a $k$ where $\Delta' > 0$ only for oblique modes
Only small $k$ modes have angular dependence

- For large $k$: $\Delta'_H \rightarrow -2k$, but $\Delta'_D \rightarrow -k(2 + \mu^2)$
- The angular dependence for large $k$ in Daughton’s $\Delta'$ is incorrect
- But $\Delta' < 0$ modes are stable, so large $k$ is not so important anyway
Boundary Layer Theory: Inner Region

- Near the resonant point, $x - x_s \equiv \xi \ll 1$, we assume $\partial_x^2 = \partial_\xi^2 \gg k_y^2$
- Expand $F$ to linear order about the resonant point:
  \[ F \simeq F'(x_s)(x - x_s) \equiv \alpha \xi \]
- Coppi et al. Sov. J. Plasma Phys. 2, 533 (1976) show:
  \[ \Delta' = -\frac{\pi}{8} (S\alpha)^{1/3} \Lambda^{5/4} \frac{\Gamma[(\Lambda^{3/2} - 1)/4]}{\Gamma[(\Lambda^{3/2} + 5)/4]} \]
  where $\Lambda \equiv \gamma S^{1/3} \alpha^{-2/3}$
- Equating this with the outer $\Delta'$ determines the growth rate
- In the large $k$ limit (the constant-$\psi$ regime), $\Lambda \ll 1$
  \[ \gamma = \left[ \frac{\Gamma(1/4)}{2\pi\Gamma(3/4)} \right]^{4/5} S^{-3/5} \alpha^{2/5} \Delta'^{4/5} \]
- In the small $k$ limit (the nonconstant-$\psi$ regime), $\Lambda \to 1^-$
  \[ \gamma = \alpha^{2/3} S^{-1/3} - \frac{2\sqrt{\pi} \alpha}{3 \Delta'} \]
Linear growth rate for oblique tearing modes

- Constant-$\psi$ and nonconstant-$\psi$ tearing mode growth rates:

\[
\gamma \tau_A \sim \begin{cases} 
S^{-3/5}(k\lambda)^{-2/5}(1 - \mu^2)^{2/5}(1 + \mu^2 - k^2\lambda^2)^{4/5}, & k\lambda S^{1/4} \gg 1 \\
S^{-1/3}(k\lambda)^{2/3}(1 - \mu^2)^{2/3}, & k\lambda S^{1/4} \ll 1 
\end{cases}
\]

- Reduces to conventional solutions for parallel modes: $\theta = \mu = 0$

- $S$ is Lundquist number based on current sheet width: $S = 4\pi \lambda V_A/(c^2 \eta)$

- $\tau_A$ is Alfvén time based on current sheet width: $\tau_A = \lambda/V_A$

- Peak growth rate is at the intersection of the two branches

\[
k_{\max} \lambda = S^{-1/4}(1 - \mu^2)^{-1/4}(1 + \mu^2)^{3/4}
\]

where

\[
\gamma_{\max} \tau_A = S^{-1/2}(1 - \mu^4)^{1/2}
\]

- Note that growth rates scale with $S$ to a negative exponent
Linear growth rate for oblique plasmoids

- Using the Sweet-Parker aspect ratio: $\lambda = \delta_{\text{SP}} = LS_L^{-1/2}$
- $S_L$ is the Lundquist number based on current sheet length: $S_L = 4\pi LV_A/(c^2\eta)$
- Plasmoid growth rates is then:
  \[
  \frac{\gamma}{\Gamma_o} \sim \begin{cases} 
  S_L^{2/5} \kappa^{-2/5} (1 - \mu^2)^{2/5} (1 + \mu^2 - \kappa^2/S_L)^{4/5} \\
  \kappa^{2/3} (1 - \mu^2)^{2/3}
  \end{cases}
  \]
  where $\Gamma_o = V_A/L$ and $\kappa = kL$
- Peak growth rate is at the intersection of the two branches
  \[
  \kappa_{\text{max}} \simeq S_L^{3/8} (1 - \mu)^{-1/4} (1 + \mu)^{3/4}
  \]
  where
  \[
  \frac{\gamma_{\text{max}}}{\Gamma_o} \simeq S_L^{1/4} (1 - \mu^4)^{1/2}
  \]
- Plasmoid growth rates scale with $S_L$ to a positive exponent
- Constant-$\psi$ branch: oblique modes ($\theta \neq 0$) most unstable
- Nonconstant-$\psi$: Parallel modes ($\theta = 0$) most unstable
BL theory accurately captures growth rate

- Numerical solutions of the linear RMHD equations (circles)
- Red dashed lines show the boundary layer theory
- Blue uses BL theory with $\Delta' = 2/(k\lambda)$ (small $k$ limit)
$k$ Dependance of Growth Rate for Fixed Angles

- Normal modes are most unstable in constant-$\psi$ branch
- Oblique modes most unstable in nonconstant-$\psi$ branch
- $\bar{B}_{yo}/B_{zo} = 0.1$ in all plots shown
Angular Dependence of Growth Rate, fixed $S_L$

- For a fixed $S_L = 10^8$, vary the wavenumber $\kappa$
- Most unstable mode is normal at small $\kappa$ (nonconstant-\(\psi\))
- Can have only oblique modes be unstable in constant-\(\psi\) branch

\[
\kappa = 1 \times 10^3, \quad 5 \times 10^3, \quad 9 \times 10^3, \quad 1 \times 10^4
\]

$S_L = 10^8$
Angular Dependence of Growth Rate, fixed $\kappa$

- For fixed $\kappa = 1 \times 10^4$, vary $S_L$
- Reach different branches as $S_L$ is varied
- Stable for $\mu > 1$ ($\theta = \bar{B}_{yo}/B_{zo} = 0.1$)
Contours of constant growth rate for $S_L = 10^8$

- Most unstable angle $|\theta| = (\tilde{B}_{yo}/B_{zo})\sqrt{(1 + \kappa^2/S_L)/3}$ in constant-$\psi$ branch
\( \phi_1 \) Symmetry Breaks at Oblique Angles

- Eigenfunctions computed numerically from the linear RMHD equations
- Resonant point given by \( x_s/\lambda \simeq -\arctanh(\theta B_z/\bar{B}_y) \)

\[ S_L = 1 \times 10^8, \kappa = 1 \times 10^3 \]

(a)
$\psi_1$ Flattens for $x > x_s$ for Oblique Modes

- Obliqueness breaks symmetry in the stream function too
- Recall: $S_L = 1 \times 10^8$, $\kappa = 1 \times 10^3$, $\theta = 0, 0.025, 0.025, 0.09$
Constant Flux Surfaces Show Asymmetries

- $\kappa = 10^3$, $S_L = 10^8$, $\bar{B}_{yo}/B_{zo} = 0.1$, and $\psi = \psi_o + 0.1\psi_1$
- $\theta = 0$ (top row), $\theta = 0.05$ radians (bottom row)
Summary

• Extended conventional tearing mode theory to account for oblique modes
• Calculated dispersion relation using Harris equilibrium and RMHD
• Primary difference with 2D theory is the location of the resonant point
  \[ x_s = -\lambda \text{arctanh} \left( \frac{k_z B_{zo}}{k_y \bar{B}_{yo}} \right) \]
• Instability requires
  \[ |\theta| < \arctan(\bar{B}_{yo}/B_{zo}) \]
  or else there is no resonant point
• Normal modes are most unstable in nonconstant-\(\psi\) branch
• Oblique modes are most unstable in constant-\(\psi\) branch
• Most unstable wavenumber is a normal mode