Radiation from an accelerated charge

• q is initially at rest at P
• q is accelerated and moves from P to P’
• After some time has passed, the electric field at point ★ is changed by an amount $E_{rad}$
• +q at rest at O; E lies along line OL
• Q given uniform accel. a along Oy for a short time interval $\tau$. At the end of this short interval q will be at P$_1$ and will be moving with velocity $v = a \tau$, where $v << c$.
• After an interval $t >> \tau$, q will reach P$_2$, where P$_1$P$_2 = v t$.

- The "kink" in the electric field line OL, due to the accel. of q, moves outward along OL with velocity c. At the end of the interval $\tau + t$, it will have reached N.
- After passing P$_1$, q moves with uniform velocity v and a point on the electric field line which leaves q at P$_1$ has moved uniformly outward and the line is now in the position P$_2$M, almost parallel to OL; also OP$_1$ and P$_1$P$_2$ are $<<$ compared to the radii of the two spheres, which are separated by an amount $d = c \tau$. 
The outer spherical shell moves outward at $c$. In front of this shell, the E field is stationary, while behind it, there is a moving field.

A disturbed field moves outward to change the field due to the acceleration. The disturbed field has radial and tangential components—$E_r$ and $E_t$.

$$\frac{E_t}{E_r} = \frac{MN'}{MM'}$$

$$MM' = c \tau$$

$$MN' = OP_2 \sin(\theta) \approx vt \sin(\theta), \text{ since } t >> \tau$$
\[ \frac{E_t}{E_r} = \frac{vt \sin(\theta)}{c \tau}, \text{ but } E_r \approx \frac{q}{4\pi\varepsilon_0 r^2}, \]

\[ \Rightarrow E_t = \frac{vt \sin(\theta)}{c \tau} \cdot \frac{q}{4\pi\varepsilon_0 r^2}, \]

but \( t \approx r/c, \) and \( v/\tau = a, \)

\[ \Rightarrow E_t = \frac{qa \sin(\theta)}{4\pi\varepsilon_0 c^2 r} \]

- This is the radiated electric field due to the accelerated charge. Note that if \( a = 0, \) \( E_t = 0 \Rightarrow \text{an accelerated charge radiates electromagnetic waves}. \)
- \( \theta \) is the angle between the velocity of the charge and the observer; \textit{there is no radiation in the forward direction} \((\theta = 0)\).
The magnetic field can be obtained simply by realizing that at large distances the radiation fields must conform to plane waves, so that \( B = \frac{E}{c} \), so with

\[
E_{rad} = \frac{qa \sin(\theta)}{4\pi\varepsilon_0 c^2 r}, \quad B_{rad} = \frac{1}{c} \frac{qa \sin(\theta)}{4\pi\varepsilon_0 c^2 r}.
\]

The Poynting vector is then:

\[
S = \frac{E_{rad} B_{rad}}{\mu_0} = \left( \frac{qa \sin(\theta)}{4\pi\varepsilon_0} \right)^2 \frac{1}{c^3 r^2}.
\]

The angular distribution of the radiation is \( \propto \sin^2(\theta) \).
Total Radiated power

The total power radiated by the accelerated charge is given by the integral of $S$ over a sphere centered at the instantaneous position of the charge.

$$P = \int S \cdot da = \left(\frac{qa}{4\pi \varepsilon_o}\right)^2 \frac{1}{c^3} \int_{0}^{\pi} \frac{\sin^2(\theta)}{r^2} \cdot 2\pi r^2 \sin(\theta) d\theta$$

$$= \left(\frac{qa}{4\pi \varepsilon_o}\right)^2 \frac{2\pi}{c^3} \int_{0}^{\pi} \sin^3(\theta) d\theta = \left(\frac{qa}{4\pi \varepsilon_o}\right)^2 \frac{2\pi}{c^3} \cdot \frac{4}{3}.$$  

Writing $\frac{1}{\varepsilon_o c^2} = \mu_o$, this can be expressed as

$$P = \frac{2}{3} \frac{\mu_o q^2 a^2}{4\pi c}.$$  

This is the famous Larmor power formula.